Recovering time-dependent inclusion in heat-conductive bodies using a dynamical probe method

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RECOVERING TIME-DEPENDENT INCLUSION IN HEAT CONDUCTIVE BODIES BY A DYNAMICAL PROBE METHOD

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Abstract. We consider an inverse boundary value problem for the heat equation $\partial_t v = \mathrm{div}_x(\gamma \nabla_x v)$ in $(0,T)\times\Omega$, where Ω is a bounded domain of $\mathbb{R}^3$, the heat conductivity $\gamma(t,x)$ admits a surface of discontinuity which depends on time and without any spatial smoothness.

The reconstruction and, implicitly, uniqueness of the moving inclusion, from the knowledge of the Dirichlet-to-Neumann operator, is realised by a dynamical probe method based on the construction of fundamental solutions of the elliptic operator $-\Delta + \tau^2\cdot$, where τ is a large real parameter, and a couple of inequalities relating data and integrals on the inclusion, which are similar to the elliptic case. That these solutions depend not only on the pole of the fundamental solution, but on the large parameter τ also, allows the method to work in the very general situation.

Key words. Inverse problem, Dirichlet-to-Neumann map, heat probing.

AMS subject classifications. 35R30, 35K05.

1. Introduction.

1.1. Inverse heat conductivity problem. Let Ω be a bounded domain in $\mathbb{R}^3$, with lipschitzian boundary $\Gamma = \partial\Omega$, and consider the following initial boundary value problem

$$\left\{\begin{array}{rcl} \partial_t v &=& \mathrm{div}_x(\gamma\nabla_x v) \quad \text{in} \quad \Omega_T = (0,T)\times\Omega, \\ v &=& f \quad \text{on} \quad \Gamma_T = (0,T)\times\Gamma, \\ v\big|_{t=0} &=& v_0 \quad \text{on} \quad \Omega, \end{array}\right. \tag{1.1}$$

where $\gamma = \gamma(t,x) \in W^{1,\infty}((0,T);L^\infty(\Omega))$ with the following properties:

(C-γ) *There exist a positive function* $(t,x)\mapsto k(t,x)$ *and, for all* $t\in[0,T]$*, a non empty open set* $D(t)\subset\Omega$, such that

- $\frac{1}{C} \le k \le C \quad \text{in} \quad D_T := \cup_{[0,T]}\{t\}\times D(t)$ for some $C>1$,
- $k-1$ has a constant sign in D_T,
- $$\gamma(t,x) = \left\{\begin{array}{ll} 1 \quad \text{if} \quad x\notin D(t), \\ k(t,x) \quad \text{if} \quad x\in D(t). \end{array}\right.$$

We don't assume any smoothness neither on $D(t)$ *nor that* $\partial D(t)\cap\Gamma = \emptyset$.

Our main purpose is to study discontinuous perturbations, however, we allow $\gamma(t,\cdot)$ to be continuous. Hence, we impose the following assumption.

(C-D) $\inf_{x\in K}|k(t,x)-1| > 0$ *for any compact set* $K\subset D(t)$, for all $t\in[0,T]$.

We shall consider a large parameter $\tau>0$ and allow the initial data $v_0(x)$ to depend on τ, under the following condition:

(C-0) There exist τ-independent positive constants C, l_0 such that $\|v_0\|_{L^2(\Omega)} \le Ce^{\tau l_0}$, for all τ.

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Physically, the region $D(t)$ corresponds to some inclusion in the medium with heat conductivity different from that in the background domain Ω. The problem we address in this paper is to determine $D(t)$ by using the knowledge of the Dirichlet-to-Neumann map (D-N map) :

$$\Lambda_{\gamma,v_0} : f \mapsto \partial_\nu v(t,x), \quad (t,x) \in \Gamma_T,$$

where $v = \mathcal{V}(\gamma; f)$ denotes the unique solution of (1.1), ν is the outer unit normal to Γ, and $\partial_\nu = \dfrac{\partial}{\partial \nu} = \nu \cdot \nabla_x$. In physical terms, $f = f(t,x)$ is the temperature distribution on the boundary and $\Lambda_{\gamma,v_0}(f)$ is the resulting heat flux through the boundary.

The above inverse boundary value problem is related to nondestructive testing where one looks for anomalous materials inside a known material.

To clarify our purpose, we remind briefly Ikehata's probe method for the elliptic inverse problem.

1.2. The elliptic situation. In the probe method for the well-known elliptic situation, Problem (1.1) is replaced by

$$\left\{ \begin{array}{rcll} \mathrm{div}_x\,(\gamma \nabla_x v) & = & 0 & \text{in} \quad \Omega, \\ v & = & f & \text{on} \quad \Gamma, \end{array} \right. \tag{1.2}$$

D_T is replaced by an open set $D \subset \Omega$. The Dirichlet-to-Neumann operator Λ_γ is a mapping: $H^{\frac{1}{2}}(\Gamma) \ni f \mapsto \partial_\nu v \in H^{-\frac{1}{2}}(\Gamma)$, where v is the unique solution of (1.2). The probe method (see [9]) starts by considering the fundamental solution $h_0(x) = \frac{1}{4\pi|x-y|}$ of $-\Delta h_0 = \delta_y$, with pole $y \in \Omega$. Then, one approximates h_0 outside a needle $\Sigma \subset \overline{\Omega}$ with one end on Γ and the other one being y by a sequence $\{h_j\}_{j\geq 1}$ such that $-\Delta_x h_j = 0$ in Ω, and estimates $\int_D |\nabla h_j(x)|^2 dx$ (or $\int_D |\nabla h_0(x)|^2 dx$) thanks to the following couple of inequalities:

$$\frac{1}{C}\int_D |\nabla h_j(x)|^2 \leq \left| \int_\Gamma (\Lambda_\gamma(h_j|_\Gamma) - \partial_\nu h_j) h_j|_\Gamma \right| \leq C \int_D |\nabla h_j(x)|^2, \tag{1.3}$$

for some $C > 1$ which does not depend on h_j. However, in the parabolic situation, inequalities like (1.3) are unclear, except in the static case $D(t) = D(0)$, for a reduced class of functions as $h(t,x) = e^{\tau^2 t} p(x)$, where $\tau > 0$ is a large parameter and p satisfies $-\Delta p + \tau^2 p = 0$ in Ω. See [7], and [8, 10] also for a similar approach.

In our work we built fundamental solutions for parabolic operators which slightly differs from the heat operator $\mathcal{L}_1 \equiv \partial_t - \Delta_x$ or to its adjoint $\mathcal{L}_1^* \equiv -\partial_t - \Delta_x$. Then, we obtain a couple of inequalities corresponding to (1.3), and, thanks to that, we run the probe method for the reconstruction of D_T.

1.3. Special fundamental solutions. Let $\tau > 0$, $y \in \mathbb{R}^3$, we consider the following function

$$p_{\tau,y}(x) := \frac{1}{r} e^{-\tau r}, \quad r = |x-y|.$$

It satisfies $p_{\tau,y} \in L^2(\mathbb{R}^3)$ and

$$-\Delta p_\tau + \tau^2 p_\tau = \delta(x-y) \text{ in } \mathbb{R}^3. \tag{1.4}$$

Thus, the functions $h_\tau(t,x) = e^{\tau^2 t} p_\tau(t,x)$ and $h_\tau^*(t,x) = e^{-\tau^2 t} p_\tau(t,x)$ are respectively solutions to the following parabolic equations

$$\mathcal{L}_1 h := \partial_t h - \Delta_x h = \delta(x-y), \quad \mathcal{L}_1^* h^* := -\partial_t h^* - \Delta_x h^* = \delta(x-y). \tag{1.5}$$

Of course, since the right term in Equalities (1.5) is not $\delta(t-s)\otimes\delta(x-y)$, our function h_τ differs from the usual fundamental solution $G_{s,y}(t,x) = \frac{\chi_{(s,+\infty)}(t)}{(4\pi(t-s))^{3/2}} e^{-\frac{|x-y|^2}{4(t-s)}}$, with parameter $(s,y) \in \mathbb{R}\times\mathbb{R}^3$, considered in [1, 2, 4].

Let Ω' be any smooth bounded domain containing $\overline{\Omega}$. Let Σ be an finite or infinite globally lipschitzian curve in Ω' with parameter $y = y(t)$, $t \in \mathbb{R}$, such that $y(t) \notin \overline{\Omega}$ for $t \leq 0$. We shall use the restriction $\Sigma|_{-1\leq t\leq T+1}$ only, which plays the role of a Ikehata's needle, but starts at point $y(-1)$ outside $\overline{\Omega}$ and can be self-intersecting. We put

$$p_{\tau,y(t)} =: p_{\tau,\Sigma}(t,x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3,$$

and replace the functions h_τ,h_τ^*, respectively by solutions of

$$(\mathcal{L}_1 + \tau\eta(t))U(t,x) = \delta(x - y(t)) \quad \text{in} \quad \mathbb{R}\times\Omega, \tag{1.6}$$
$$(\mathcal{L}_1^* + \tau\eta(t))U^*(t,x) = \delta(x - y(t)) \quad \text{in} \quad \mathbb{R}\times\Omega, \tag{1.7}$$

where functions $\eta(t)$ are not necessarily the same in (1.6) and (1.7), but belong to $L^\infty(\mathbb{R})$ with a τ- independent upper bound:

$$|\eta(t)| \leq \mu, \quad t \in \mathbb{R}, \tag{1.8}$$

for some $\mu > 0$ that we shall precise later. (By $\dot{q}$ we denote the derivative of a function q according to the time variable t).

The solutions U and U^* we look for should be respectively written $U_{\tau,T,\Sigma}$, $U^*_{\tau,T,\Sigma}$, but we simply denote them by U_τ, U_τ^*. We construct them so that they have the following form:

$$U_\tau(t,x) = e^{\tau^2 t} u_\tau(t,x), \quad U_\tau^*(t,x) = e^{-\tau^2 t} u_\tau^*(t,x) \quad \text{in} \quad \Omega_T,$$

with functions $u_\tau(t,x)$, $u_\tau^*(t,x)$ sufficiently close to $p_{\tau,\Sigma}(t,x)$ in the sense below.

LEMMA 1.1. *There exists a function $\varphi(t,x;\tau) \in C([0,T]; H^1_{loc}(\mathbb{R}^3))$ satisfying the following points.*

$$\varphi(t,y(t);\tau) = 1, \quad t \in \mathbb{R}, \quad \tau \geq 0. \tag{1.9}$$

There exists $\tau_1(\mu)$ such that for all $\tau > \tau_1$, we have

$$\frac{1}{C(\mu)} \leq \varphi(t,x;\tau) \leq C(\mu) \quad (t,x) \in \mathbb{R}\times\Omega, \tag{1.10}$$
$$|\nabla_x \varphi(t,x;\tau)| \leq C(\mu), \quad (t,x) \in \mathbb{R}\times\Omega, \tag{1.11}$$

for some $C(\mu) > 0$ which does not depend on τ or $(t,x) \in \mathbb{R}\times\Omega$. The function

$$U_\tau(t,x) := e^{\tau^2 t} u_\tau(t,x) \equiv e^{\tau^2 t} \varphi(t,x;\tau) p_{\tau,\Sigma}(t,x) \tag{1.12}$$

satisfies (1.6) for all $\tau > 0$.

Remark 1. *Similarly we construct* $\varphi^*(t,x;\tau) \in C([0,T];H^1_{loc}(\mathbb{R}^3))$ *such that the function*

$$U^*_\tau(t,x) = e^{-\tau^2 t} u^*_\tau(t,x) = e^{-\tau^2 t}\varphi^*(t,x;\tau) p_{\tau,\Sigma}(t,x)$$

satisfies (1.7), with Relation 1.9 and Estimates (1.10), (1.11) where φ *is replaced by* φ^*.

We choose functions η as follows. Let us consider the following function

$$\kappa(t) := e^{-\tau\mu|t-\theta|}, \quad t \in \mathbb{R}, \tag{1.13}$$

where the parameters $\mu > 0$, $\theta \in [0,T]$, are independent of τ and will be precised later. For U_τ, we put simply $\eta \equiv 0$. For U^*_τ, we put

$$\eta(t) = -\frac{\dot{\kappa}}{\tau\kappa} = \mu\,\mathrm{sgn}(t-\theta).$$

(Thus each η satisfies (1.8) obviously).

1.4. Needle sequences. Let $T' \in (0,T]$, we put $\Sigma_{T'} = \{(t,y(t)),\ 0 \le t \le T'\}$. We consider needle sequences $\{U_j\}_j$, $\{U^*_{j,T'}\}_j$, respectively associated to $(\Sigma_T, U_\tau, \eta \equiv 0)$, $(\Sigma_{T'}, U^*_\tau, \eta \equiv -\frac{\dot{\kappa}}{\tau\kappa})$. They satisfy:

$$\begin{aligned}(\mathcal{L}_1 + \tau\eta(t))U_j = 0 \quad &\text{in} \quad \Omega_T,\\ U_j(0,\cdot) = U_\tau(0,\cdot) \quad &\text{in} \quad \Omega,\end{aligned}$$

$$(\mathcal{L}^*_1 + \tau\eta(t))U^*_{j,T'} = 0 \quad \text{in} \quad \Omega_{T'}, \tag{1.14}$$

$$U^*_{j,T'}(T',\cdot) = U^*_\tau(T',\cdot) \quad \text{in} \quad \Omega, \tag{1.15}$$

and, for all open set $V \subset \Omega_{T'}$ such that $\mathrm{d}(V, \Sigma_{T'}) > 0$,

$$(U_j, U^*_{j,T'}) \overset{j\to\infty}{\to} (U_\tau, U^*_\tau) \quad \text{(strongly) in} \quad (H^{1,0}(V) \cap H^{0,1}(V))^2. \tag{1.16}$$

Here, we use the notation

$$H^{s,p}(V) = \overline{C_0^\infty(V)}^{\|\cdot\|_{s,p}}, \quad s,p \ge 0,$$

with $\|u\|_{s,p} \equiv \|(t \mapsto \|u(t,\cdot)\|_{H^p(\mathbb{R}^3;dx)})\|_{H^s(\mathbb{R};dt)}$.

Remark 2. *Since* $y(0) \notin \overline{\Omega}$, *thanks to (1.16), we have*

$$U^*_{j,T'}(0,\cdot) \overset{j\to\infty}{\to} U^*_\tau(0,\cdot) \quad \textit{(strongly) in} \quad L^2(\Omega).$$

1.5. Reflected waves. Let $V_j := \mathcal{V}(\gamma, U_j|_{\Gamma_T})$ be the solution of (1.1) with data $f = U_j|_{\Gamma_T}$, put $W_j = V_j - U_j$. By denoting $\mathcal{L}_\gamma = \partial_t - \nabla_x((\gamma-1)\nabla_x$, we have

$$\mathcal{L}_\gamma W_j = \nabla_x((\gamma-1)\nabla_x U_j) \quad \text{in} \quad \Omega_T, \tag{1.17}$$
$$W_j = 0 \quad \text{on} \quad \Gamma_T, \tag{1.18}$$
$$W_j|_{t=0} = v_0 - U_\tau \quad \text{in} \quad \Omega.$$

If $\Sigma_T \cap \overline{D_T} = \emptyset$ then, in view of (1.17), we have

$$W_j \overset{j\to\infty}{\to} W_\tau \quad \text{in} \quad C([0,T], H^1(\Omega)), \tag{1.19}$$

where W_τ is the unique solution of

$$\begin{aligned} \mathcal{L}_\gamma W_\tau &= \nabla_x((\gamma-1)\nabla_x U_\tau) \quad \text{in} \quad \Omega_T, \\ W_\tau &= 0 \quad \text{on} \quad \Gamma_T, \\ W_\tau|_{t=0} &= v_0 - U_\tau \quad \text{in} \quad \Omega. \end{aligned} \tag{1.20}$$

Thus

$$V_j \overset{j\to\infty}{\to} V_\tau = U_\tau + W_\tau \quad \text{in} \quad H^{0,1}(V), \tag{1.21}$$

for all open set V such that $\overline{V} \subset \overline{\Omega_T} \setminus \Sigma_T$. The function V_τ satisfies

$$\begin{aligned} \mathcal{L}_\gamma V_\tau &= \delta(x - y(t)) \quad \text{in} \quad \Omega_T, \\ V_\tau &= U_\tau \quad \text{on} \quad \Gamma_T, \\ V_\tau|_{t=0} &= v_0 \quad \text{in} \quad \Omega. \end{aligned}$$

1.6. Pre-indicator sequence and indicator function. Let $\theta \in (0,T)$, $\mu > 0$, and put

$$\begin{aligned} I_j(\tau,\mu,\theta,T) &:= \int_{\Gamma_T} (\partial_\nu W_j)\, U^*_{j,T}\, d\sigma(x)\kappa(t)dt, \\ I_\infty(\tau,\mu,\theta,T) &:= \int_{\Omega_T} (\gamma-1)\nabla_x V_\tau \nabla_x U^*_\tau\, dx\kappa(t)dt + \int_\Omega [\kappa W_\tau U^*_\tau]_0^T\, dx, \end{aligned}$$

where κ is defined by (1.13), and $d\sigma(x)$ is the usual measure on the boundary Γ.

REMARK 3. *As opposite to dynamical probe methods in [2, 10], the indicator function I_∞ depends on the needle Σ.*

The following result ensures us that we have the knowledge of $I_\infty(\tau,\mu,\theta,T)$ from the Cauchy data of (1.1) when Σ_T does not touch D_T.

LEMMA 1.2. *Assume that $\Sigma_T \cap \overline{D_T} = \emptyset$. Then, for all $\theta \in [0,T]$, $\mu, \tau \geq 0$, we have*

$$I_j(\tau,\mu,\theta,T) \overset{j\to\infty}{\to} I_\infty(\tau,\mu,\theta,T) \in \mathbb{R}. \tag{1.22}$$

1.7. Main theorems. The following theorem separates the cases $\Sigma_T \cap \overline{D_T} = \emptyset$ and $\Sigma_T \cap \overline{D_T} \neq \emptyset$. In the second case, we can put

$$T^* := \min\{T' \in [0,T];\ \Sigma_{T'} \cap \overline{D_{T'}} \neq \emptyset\} > 0.$$

We put also

$$\mu_{T',\theta} := l_0 \max\left(\frac{1}{\theta}, \frac{1}{(T'-\theta)}\right), \quad 0 < \theta < T' \leq T,$$

where l_0 is the constant in (C-0).

By $\mathrm{d}(y,X)$ we denote the "distance" from the point y to the set X, and by $\mathrm{d}(Y,X)$ the "distance" from the set Y to the set X:

$$\mathrm{d}(Y,X) = \inf_{y\in Y} \mathrm{d}(y,X), \quad \mathrm{d}(y,X) = \inf_{x\in X} |x-y|.$$

Theorem 1. *Assume (C-D) and let $\theta \in (0,T)$. Assume that $\Sigma_T \cap \overline{D_T} = \emptyset$. Then there exists $\mu_1 > \mu_{T,\theta}$ such that, for all $\mu > \mu_1$, there exist $C(\mu) > 1$, $\tau_1 > \mu_1$ such that, under assumption (C-0), we have*

$$\begin{aligned}\frac{1}{C(\mu)} \int_{\Omega_T} |\gamma - 1|\, |\nabla_x p_{\tau,\Sigma}|^2 dx\kappa dt \leq |I_\infty(\tau,\mu,\theta,T)| \leq \\ C(\mu) \int_{\Omega_T} |\gamma - 1|\, |\nabla_x p_{\tau,\Sigma}|^2 dx\kappa dt,\end{aligned} \tag{1.23}$$

Theorem 2. *Assume (C-D). Then the following points hold.*

- *Let $\theta \in (0,T)$. For all $\mu > \mu_1$, under assumptions (C-0), $\Sigma_T \cap \overline{D_T} = \emptyset$, we have*

$$\lim_{\tau\to\infty} \frac{1}{\tau} \ln |\, I_\infty(\tau,\mu,\theta,T)| = -2d(y(\theta), D(\theta)) < 0. \tag{1.24}$$

- *under assumptions (C-0), $\Sigma_T \cap \overline{D_T} = \emptyset$, we have*

$$\limsup_{\theta\in(0,T)} \lim_{\mu\to\infty} \lim_{\tau\to\infty} \frac{1}{\tau} \ln |\, I_\infty(\tau,\mu,\theta,T)| = -2d(\Sigma_T, D_T) < 0. \tag{1.25}$$

- *Assume that $\Sigma_T \cap \overline{D_T} \neq \emptyset$.*

Then, under assumption (C-0), we have

$$\lim_{T'\nearrow T^*} \limsup_{\theta\in(0,T')} \lim_{\mu\to\infty} \lim_{\tau\to\infty} \frac{1}{\tau} \ln |\, I_\infty(\tau,\mu,\theta,T')| = 0. \tag{1.26}$$

So we can detect if Σ_T touches D_T or not. Consider the function

$$F(T') = \limsup_{\theta\in(0,T')} \lim_{\mu\to\infty} \lim_{\tau\to\infty} \lim_{j\to\infty} \frac{1}{\tau} \ln |\, I_j(\tau,\mu,\theta,T')|, \quad 0 \leq T' \leq T,$$

defined in $[0,T^*[$. The problem is that T^* is defined from F itself. The following result gives a strategy to determine T^*.

Lemma 1.3. *There exists $\delta > 0$ such that if $T' < T^*$ then $T' < T' + |F(T')|/\delta < T^*$.*

In practice, if we have the a priori knowledge of δ, we can determine T^* as the limit of the sequence

$$(t_n)_{n\in\mathbb{N}} : \quad t_0 = 0; \quad t_{n+1} = t_n + |F(t_n)|/\delta,\ n \geq 0.$$

If $t_{n+1} > T$ for some n, this indicates that Σ_T does not touch D_T. We write $T^* = T + 0$ in this case.

Corollary 1.1. *From the knowledge of Λ_{γ,v_0} we can compute :*
(1) the maximal time T^* of a given curve Σ.
(2) the connected components of $\overline{\Omega_T} \setminus \overline{D_T}$ that touch $\{0\} \times \Gamma$.

In particular, the connected components of $\overline{\Omega_T} \setminus \overline{D_T}$ that touch $\{0\} \times \Gamma$ are completly characterized from Λ_{γ,v_0}.

Let us make some remarks on Theorem 1, Corollary 1.1.

Remark 4. *- We emphazise the fact that we don't make assumption on $D(t)$, $0 \leq t \leq T$.*
- From the knowledge of Λ_{γ,v_0}, we get the information not only on $d(\Sigma_{T'}, D(T'))$, when $T' < T^*$, but on $d(y(\theta), D(\theta))$, $\theta \in (0, T')$, too.

1.8. Literature review. Many articles solve a version of the Calderón inverse problem for the heat equation. The biggest part of them assume that the unknown coefficient γ, or the unknown inclusion D, do not depend on time t: see for example [8, 10]

But they are very few results concerning the time dependent situation $D = D(t)$. The authors in [3] proved uniqueness of D_T under the assumption that the inclusion $D(t)$ is x-lipchitzian for all t. They used a proof by contradiction and is not constructive at all. A more recent paper for a similar question is [11].

A reconstruction method by a dynamical probe method is performed in [2], but it works for the one dimensional spatial space only. In this case, an easier way is to put as input the trace of plane waves $U_\tau(t, x) = e^{\tau^2 t + \tau x}$ at the spatial boundary, then to compute the solution $V_\tau(t, x)$ by using an ansatz and an energy estimate: see [5]. This approach works in the x-multidimensional case if $D(t)$ is a convex set: see [6].

With the aim of working the dynamical probe method in the x-multidimensional case with, the authors of [4] computed the reflected solution at points (t, x) close to the lateral boundary of D_T. But this way is painful enough to assume that $D(t) = D$ does not depend on t. Moreover, since the data these authors consider are the traces on Γ_T of Runge approximations of the fundamental solution $G_{s,y}$ of the heat equation, this method requires some regularity of ∂D (more precisely, they assumed that ∂D is of class $C^{1,\alpha}$, for some $\alpha \in (0, 1)$).

In [1] the stability of the operator $\gamma \mapsto \Lambda_{\gamma,0}$ is quantified, under the assumption that D_T belongs to some class $\mathcal{K}$ contained in $L^\infty((0, T); W^{2,\infty}(\Omega)) \cap W^{1,\infty}((0, T); W^{1,\infty}(\Omega))$.

We think that our approach, based on special fundamental solutions and inequalities (1.23), would be able to extend the frameworks of many of these articles.

1.9. Outline of the paper. The paper is organized as follows. Section 2 is devoted to the proofs and is devided in several subsections. In subsection 2.1, we give basic estimates for integrals on D_T with weight functions as $(p_{\tau,\Sigma}\kappa)(t,x)$. In subsection 2.2, we build the functions U_τ, U_τ^* by proving Lemma 1.1 of section 1.4. In subsection 2.7 we prove Lemma 1.3. In subsection 2.5 we prove Theorem 1. In subsection 2.6 we prove Theorem 2. In subsection 2.8, we build the needle sequences U_j, $U_{j,T'}$.

2. Proofs.

Notations. We consider the usual Sobolev spaces $H^m(V)$, where $V \subset \mathbb{R}^k$ is an open set, and also $H^m((T_1,T_2);H^{m'}(V)) = H^m(T_1,T_2;H^{m'}(V))$, for $m,m' \in \mathbb{Z}$. Here, H^0 is the well-know space L^2.

The formal parabolic operator is $\mathcal{L}_\gamma = \partial_t - \nabla_x \cdot (\gamma \nabla_x \cdot)$.

2.1. Basic Estimates. In this part we establish some basic estimates on W_τ etc.

Lemma 2.1. *a) Let $\mathcal{O} \subset \mathbb{R}^3$ be an non empty open set and $\tau > 0$. Then we have*

$$\int_{\mathcal{O}} |p_{\tau,y}(x)|^2 dx \le C \frac{1}{\tau\, d(y,\mathcal{O})} e^{-2\tau\, d(y,\mathcal{O})}, \quad \forall y \in \mathbb{R}^3 \tag{2.1}$$

$$\int_{\mathcal{O}} |\nabla_x p_{\tau,y}(x)|^2 dx \le C\tau \left(1 + \frac{1}{\tau\, d(y,\mathcal{O})}\right) e^{-2\tau\, d(y,\mathcal{O})}, \quad \forall y \notin \overline{\mathcal{O}}. \tag{2.2}$$

b) Let $\mathcal{O} \subset \mathbb{R}^3$ is a non empty open. Let $y \notin \overline{\mathcal{O}}$ and $d > d(y,\mathcal{O})$. Then there exists $C(\mathcal{O},y,d) > 0$, such that

$$\int_{\mathcal{O}} |\nabla_x p_{\tau,y}(x)|^2 dx \ge C(\mathcal{O},y,d)\tau^2 e^{-2\tau d}, \quad \forall \tau \ge 0. \tag{2.3}$$

Proof. a) We check easily:

$$\begin{aligned}\int_{\mathcal{O}} |p_{\tau,y}(x)|^2 dx &= \int_{\mathcal{O}} \frac{1}{(4\pi)^2 r^2} e^{-2\tau r} dx \le \int_{r>\mathrm{d}(y,\mathcal{O})} \frac{1}{4\pi} e^{-2\tau r} dr \\ &\le \frac{1}{\tau} \int_{s>\tau \mathrm{d}(y,\mathcal{O})} e^{-2s} ds = (2\tau \mathrm{d}(y,\mathcal{O}))^{-1} e^{-2\tau \mathrm{d}(y,\mathcal{O})}.\end{aligned}$$

Hence (2.1). Similarly we have

$$\begin{aligned}\int_{\mathcal{O}} |\nabla_x p_{\tau,y}(x)|^2 dx &= \int_{\mathcal{O}} \frac{(\tau r+1)^2}{(4\pi)^2 r^4} e^{-2\tau r} dx \le \int_{r>\mathrm{d}(y,\mathcal{O})} \frac{(\tau r+1)^2}{4\pi r^4} e^{-2\tau r} r^2 dr \\ &\le \frac{\tau}{4\pi} \int_{s>\tau\mathrm{d}(y,\mathcal{O})} \frac{(s+1)^2}{s^2} e^{-2s} ds \\ &\le C\tau \left(1 + \frac{1}{\tau \mathrm{d}(y,\mathcal{O})}\right) e^{-2\tau \mathrm{d}(y,\mathcal{O})}.\end{aligned}$$

b) Let us prove (2.3). Observe that the open set $\mathcal{O}_d := \{x \in \mathcal{O};\ |x-y| < d\}$ is non empty. Hence we get

$$J(y) := \int_{\mathcal{O}} |\nabla_x p_{\tau,y}(x)|^2 dx \ge \int_{\mathcal{O}_d} \frac{(\tau r+1)^2}{(4\pi)^2 r^4} e^{-2\tau d} dx \ge C\tau^2 e^{-2\tau d},$$

with $C = \int_{\mathcal{O}_d} \frac{1}{(4\pi r)^2} dx > 0$.

REMARK 5. *It is clear that if $y \in \mathcal{O}$, or if $\partial\mathcal{O}$ is lipschitz and $y \in \partial\mathcal{O}$, then*

$$\int_{\mathcal{O}} |\nabla_x p_{\tau,y}(x)|^2 dx = +\infty.$$

LEMMA 2.2. *Assume that $\Sigma_T \cap \overline{D_T} = \emptyset$. Let $\theta \in [0,T]$. Let $d > d(y(\theta), D(\theta))$. Then, there exists $\mu_1 > 0$, $\tau_1 > 0$ such that, for all $\mu > \mu_1$, $\tau > \tau_1$, we have*

$$\int_{\Omega_T} |\gamma - 1|\, |\nabla_x p_{\tau,\Sigma}(x)|^2 dx\kappa(t)dt \leq \frac{C}{\mu}\big(1 + \frac{1}{\tau d(y(\theta), D(\theta))}\big)e^{-2\tau d(y(\theta),D(\theta))}, \quad (2.4)$$

$$\int_{\Omega_T} |\gamma - 1|\, |\nabla_x p_{\tau,\Sigma}(x)|^2 dx\kappa(t)dt \geq C\frac{\tau}{\mu}e^{-2\tau d}, \quad (2.5)$$

for some $C > 0$ which does not depend on μ or τ. Proof.

a) Thanks to (2.2) we have

$$\begin{aligned} I &:= \int_{\Omega_T} |\gamma - 1|\, |\nabla_x p_{\tau,\Sigma}(x)|^2 dx\kappa(t)dt \\ &\leq C\tau \int_0^T \Big(1 + \frac{1}{\tau \mathrm{d}(y(t), D(t))}\Big)e^{-2\tau \mathrm{d}(y(t),D(t))}\kappa(t)dt. \end{aligned}$$

By assumption on D_T and Σ, we have

$$\mathrm{d}(y(\theta), D(\theta)) + \delta|t-\theta| \geq \mathrm{d}(y(t), D(t)) \geq \mathrm{d}(y(\theta), D(\theta)) - \delta|t-\theta|, \quad t \in [0,T],$$

for some $\delta > 0$ depending on Σ_T and D_T only. Fix $\mu_1 > 2\delta$ and let $\mu \geq \mu_1$. Put $\varepsilon = \frac{\mathrm{d}(y(\theta),D(\theta))}{\sqrt{\tau}}$. We have

$$\begin{aligned} I \leq\, & C\tau e^{-2\tau \mathrm{d}(y(\theta),D(\theta))}\Big(1 + \frac{1}{\tau(\mathrm{d}(y(\theta), D(\theta)) - \delta\varepsilon)}\Big)\int_{|t-\theta|<\varepsilon} e^{\tau(2\delta-\mu)|t-\theta|}dt \\ & + Te^{-2\tau \mathrm{d}(y(\theta),D(\theta))}e^{\tau\varepsilon(2\delta-\mu)}\Big(1 + \frac{1}{\tau(\mathrm{d}(\Sigma_T, D_T)}\Big)dt \\ \leq\, & e^{-2\tau \mathrm{d}(y(\theta),D(\theta))}\Big\{\frac{C}{\mu - 2\delta}\big(1 + \frac{2}{\tau \mathrm{d}(y(\theta), D(\theta))}\big) + \mathcal{O}(e^{-\delta'\sqrt{\tau}})\Big\}. \end{aligned}$$

for some $\delta' > 0$. Estimate (2.4) is proved.

b) Put

$$J(t) := \int_{\Omega} |\gamma(t,x) - 1|\, |\nabla_x p_{\tau,y(t)}(x)|^2 dx.$$

Thanks to assumption (C-D) we have

$$J(\theta) \geq \delta \int_{\Omega} |\nabla_x p_{\tau,y(\theta)}(x)|^2 dx,$$

with $\delta = \liminf_{x\in\overline{\mathcal{O}}} |\gamma(\theta,x) - 1| > 0$. Then, thanks to (2.3), we have

$$J(\theta) \geq \delta'\tau^2 e^{-2\tau d},$$

for some $\delta' > 0$. Obviously, $J(\cdot)$ depends continuously on t, since $\gamma \in L^\infty(\Omega_T)$. Hence if $|t-\theta| < \varepsilon$ and $\varepsilon > 0$ is sufficiently small, we have

$$J(t) \geq \frac{1}{2}\delta'\tau^2 e^{-2\tau d}.$$

We then have

$$I := \int_{\Omega_T} J(t)\kappa(t)dt \geq \int_{|t-\theta|<\varepsilon} J(t)\kappa(t)dt \geq C\tau^2 e^{-2\tau d}\int_{|t-\theta|<\varepsilon}\kappa(t)dt$$
$$\geq C\frac{\tau}{\mu}e^{-2\tau d}(1-e^{-\tau\mu\varepsilon}).$$

for some $C > 0$. Taking $\mu_1 > \frac{1}{\varepsilon}$, this proves (2.5).

LEMMA 2.3. *Let $\theta \in (0,T)$. Then there exist $\delta > 0$, $\mu_1 > 0$ such that for all $\mu > \mu_1$ there exist $C(\mu) \geq 0$, $\tau_1 > 0$ such that*

$$\int_\Omega \kappa(0)|U|^2 dx \leq C(\mu)e^{-2\tau\, d(y(t),D(t))+\delta)}, \quad t \in [0,T],\ \tau > \tau_1,$$
$$U = U_\tau(0), U_\tau^*(0); \tag{2.6}$$

$$\int_\Omega \kappa(T)|u_\tau^*(T)|^2 dx \leq C(\mu)e^{-2\tau(\,d(y(\theta),D(\theta))+\delta)}, \quad \tau > 1; \tag{2.7}$$

$$\int_\Omega \kappa(0)|W_\tau(0)|^2 dx \leq C(\mu)e^{-2\tau\, d(y(t),D(t))+\delta)}, \quad t \in [0,T],\ \mu > \mu_1,\ \tau > 1; \tag{2.8}$$

$$\int_\Omega \kappa(0)|W_\tau(0)U^*(0)|dx \leq C(\mu)e^{-2\tau\, d(y(t),D(t))+\delta)}, \quad t \in [0,T],\ \tau > \tau_1. \tag{2.9}$$

Proof. a) Thanks to Lemma 1.1, (1.10), to Remark 1, to (2.1), and reminding that $y(0) \notin \overline{\Omega}$, we have

$$\int_\Omega |U|^2 dx \leq C\int_\Omega |p_{\tau,y(0)}|^2 dx \leq C\tau^{-1}e^{-2\tau d(y(0),\Omega)} \leq C'.$$

Since $\kappa(0) = e^{-\tau\mu\theta}$, we then have

$$\int_\Omega \kappa(0)|U|^2 dx \leq C'e^{-\tau\mu\theta}. \tag{2.10}$$

Choosing $\mu_1 > \sup_{(t,x)\in\Omega_T}\frac{2\mathrm{d}(y(t),x)}{\theta}$, we obtain (2.6).
b) Similarly, choosing $\mu_1 > \sup_{(t,x)\in\Omega_T}\frac{2\mathrm{d}(y(t),x)}{(T-\theta)}$, we obtain (2.7).
c) Let us remind that $W_\tau(0,x) = v_0(x) - U_\tau(0,x)$. Thanks to Assumption (C-0) and to (1.10), (1.12), (2.6), we have

$$\int_\Omega \kappa(0)|W_\tau(0)|^2 dx \leq Ce^{-\tau\mu\theta}(e^{l_0\tau} + \tau^{-1}e^{-2\tau d(y(0),\Omega)}) \leq Ce^{-\theta(\mu-\mu_{T,\theta})\tau}.$$

Choosing $\mu_1 > \mu_{T,\theta} + \sup_{(t,x)\in\Omega_T}\frac{2\mathrm{d}(y(t),x)}{\theta}$ we obtain (2.8).
d) Estimates (2.6), (2.8) and standart inequalities imply (2.9).

2.2. Construction of the special functions. Let $R > 0$ such that $\Omega \subset \{x \mid |x| < R\} = B_R$. Put $z = x - y(t)$, $u_\tau(t,x) = f(t,z)$. We have

$$\partial_t u_\tau = \partial_t f + \dot{z}\nabla_z f = \partial_t f - \dot{y}\nabla_z f,$$

$$(\partial_t - \Delta_z)f - \dot{y}(t)\nabla_z f + (\tau^2 + \tau\eta(t))f = \delta_0(z).$$

Let $\hat{f}(t,k) = (2\pi)^{-3/2}\int_{z\in\mathbb{R}^3} f(t,z)e^{-ikz}dz$ be the Fourier transform of $f(t,\cdot)$. It satisfies

$$\partial_t \hat{f} + (k^2 - ik\dot{y}(t) + \tau^2 + \tau\eta(t))\hat{f} = 1.$$

Put $\rho(t) = \int_0^t \eta(s)ds$. We then have

$$\hat{f}(t,k) = (2\pi)^{-3/2}\int_{-\infty}^{t} \exp\left(k^2(s-t) - ik(y(s)-y(t)) + \tau^2(s-t) + \tau(\rho(s)-\rho(t))\right) ds, \tag{2.11}$$

and so

$$\begin{aligned} f(t,z) &= (2\pi)^{-3/2}\int_{-\infty}^{t}\left(\int_{k\in\mathbb{R}^3} e^{k^2(s-t)+ik(z+y(t)-y(s))}dk\right)e^{\tau^2(s-t)+\tau(\rho(s)-\rho(t))}ds, \\ &= \frac{1}{8\pi^{3/2}}\int_0^\infty\left(\int_{k\in\mathbb{R}^3} e^{-k^2 s+ik(z+y(t)-y(t-s))}dk\right)e^{-\tau^2 s+\tau(\rho(t-s)-\rho(t))}ds, \\ u_\tau(t,x) &= \frac{1}{8\pi^{3/2}}\int_0^\infty e^{-\tau^2 s+\tau(\rho(t-s)-\rho(t))}e^{\frac{-(x-y(t-s))^2}{4s}}s^{-3/2}ds. \end{aligned} \tag{2.12}$$

The integral in (2.12) is convergent if and only if $x \neq y(t)$. We set

$$\varphi := \frac{u_\tau}{p_{\tau,\Sigma}}.$$

Let us prove (1.10). We put

$$\begin{aligned} g_\tau(s,t,x) &= e^{\tau(\rho(t-s)-\rho(t))+\frac{1}{4s}(y(t-s)-y(t))(2x-y(t-s)-y(t))}, \\ k_\tau(s,t,x) &= \frac{1}{8\pi^{3/2}}e^{-\tau^2 s}e^{\frac{-(x-y(t))^2}{4s}}s^{-3/2}, \end{aligned}$$

and write

$$u_\tau(t,x) = \int_0^\infty k_\tau(s,t,x)g_\tau(s,t,x)ds.$$

Let us observe that

$$p_{\tau,y(t)}(x) = \int_0^\infty k_\tau(s,t,x)ds, \quad x \in \mathbb{R}^3.$$

In fact, the right-hand side in the above relation belongs to $L^2(\mathbb{R}^3)$, with a Fourier transform corresponding to (2.11) where g is replaced by 1. Moreover, it satisfies (1.4), since it is a time-independent solution of (1.6). Hence this right-hand side is $p_{\tau,y(t)}(x)$.

Since $\rho(\cdot)$ is lipschitzian, we can fix $K \in \mathbb{R}$ such that $\tau > |K| > \|\dot\rho\|_\infty$. Let us remind that $\dot\rho = \eta$ and η satisfies (1.8). Thus the above inequalities impose $\mu < \tau$, and, in fact,

$$\mu << \tau, \quad \text{as} \quad \tau \to \infty.$$

Then, if $K > 0$, we can write, for $x \in \Omega \subset B_R$, $x \neq y(t)$,

$$\begin{aligned} k_\tau(s,t,x) g_\tau(s,t,x) &= k_{\tau-K}(s,t,x) e^{2\tau K s + K^2 s} g_\tau(s,t,x) ds \\ &\leq C(\mu,K,R) k_{\tau-K}(s,t,x). \end{aligned}$$

Hence

$$u_\tau \leq C(\mu,K,R) p_{\tau,\Sigma} \quad \text{in} \quad \Omega_T.$$

For $K < 0$ we obtain

$$u_\tau(t,x) \geq p_{\tau-K,y(t)}(x) \geq C'(\mu,K,R) p_{\tau,y(t)}(x),$$

where $C'(\mu,K,\Omega) > 0$. Thus (1.10) is proved. Let us prove (1.11). Let $x \in \Omega \subset B_R$, $x \neq y(t)$, We use the following changes of variable

$$s = \frac{r}{2\tau} e^a, \quad \tilde s := \frac{r}{2\tau} e^{-a}, \quad b = \cosh a \; (a > 0), \quad \alpha = \tau r b', \quad b' = b - 1.$$

We then have

$$\begin{aligned} u_\tau(t,x) &= \frac{1}{8\pi^{3/2}} \int_0^\infty e^{-\tau^2 s - \frac{r^2}{4s}} s^{-3/2} g_\tau(s,t,x) ds \\ &= \frac{1}{8\pi^{3/2}} (\frac{r}{2\tau})^{-1/2} \int_{-\infty}^\infty e^{-\tau r \cosh a} e^{-a/2} g_\tau(s,t,x) da \\ &= \frac{1}{8\pi^{3/2}} (\frac{2\tau}{r})^{1/2} \int_0^\infty e^{-\tau r \cosh a} (e^{-a/2} g_\tau(s,t,x) + e^{a/2} g_\tau(\tilde s,t,x)) da \\ &= \frac{1}{8\pi^{3/2}} (\frac{2\tau}{r})^{1/2} \int_0^\infty e^{-\tau r b} \Big(\cosh(a/2)(g_\tau(s,t,x) + g_\tau(\tilde s,t,x)) - \\ &\quad \sinh(a/2)(g_\tau(s,t,x) - g_\tau(\tilde s,t,x)) \Big) da \\ &\equiv u^+ - u^-, \end{aligned}$$

with

$$\begin{aligned} u_\tau^+(t,x) &= \frac{1}{8\pi^{3/2}} (\frac{2\tau}{r})^{1/2} \int_0^\infty (g_\tau(s,t,x) + g_\tau(\tilde s,t,x)) e^{-\tau r b} \cosh(a/2) da \\ &= \frac{1}{8\pi^{3/2}} (\frac{2\tau}{r})^{1/2} e^{-\tau r} \int_0^\infty (g_\tau(s,t,x) + g_\tau(\tilde s,t,x)) e^{-\tau r b'} \frac{\cosh(a/2)}{\sinh(a)} db' \\ &= \frac{\sqrt{2}}{8\pi^{3/2}} \frac{1}{r} e^{-\tau r} \int_0^\infty (g_\tau(s,t,x) + g_\tau(\tilde s,t,x)) \frac{\cosh(a/2)}{\sinh(a)} e^{-\alpha} \frac{d\alpha}{\sqrt{\tau r}} \\ &\equiv p_{\tau,y(t)} \varphi^+(t,x), \\ u_\tau^-(t,x) &= \frac{1}{8\pi^{3/2}} (\frac{2\tau}{r})^{1/2} \int_0^\infty (g_\tau(s,t,x) - g_\tau(\tilde s,t,x)) e^{-\tau r b} \sinh(a/2) da \\ &= \frac{\sqrt{2}}{8\pi^{3/2}} \frac{1}{r} e^{-\tau r} \int_0^\infty (g_\tau(s,t,x) - g_\tau(\tilde s,t,x)) \frac{\sinh(a/2)}{\sinh(a)} e^{-\alpha} \frac{d\alpha}{\sqrt{\tau r}} \\ &\equiv p_{\tau,y(t)} \varphi^-(t,x). \end{aligned}$$

Estimate (1.11) in Lemma 1.1 is a consequence of the relation $\varphi = \varphi^- + \varphi^+$ and the folllowing Lemma.

LEMMA 2.4. *We have, for $r \leq R$ and τ sufficiently large ($\tau > C(\mu, R)$),*

$$|\varphi^-(t,x)| \leq C(\mu,R)/\sqrt{\tau}, \tag{2.13}$$
$$|\varphi^+(t,x)| \leq C(\mu,R), \tag{2.14}$$
$$|\nabla_x \varphi^{\pm}(t,x)| \leq C(\mu,R). \tag{2.15}$$

Proof. We observe that $0 < \tilde{s} < 1 < s$ for $a > 0$, and

$$\begin{aligned}
\cosh(a/2) &= \sqrt{(b'+2)/2} = \sqrt{\frac{\alpha}{2\tau r} + 1},\\
\sinh(a/2) &= \sqrt{b'/2} = \sqrt{\frac{\alpha}{2\tau r}},\\
\tau(s-\tilde{s}) &= r\sinh(a) = r\sqrt{b^2-1} = r\sqrt{b'(b'+2)} = \frac{1}{\sqrt{\tau}}\sqrt{\alpha}\sqrt{\frac{\alpha}{\tau} + 2r},\\
&\leq \frac{\alpha}{\tau} + \sqrt{\frac{2r}{\tau}}\sqrt{\alpha},\\
\tau(s+\tilde{s}) &= r\cosh(a) = rb = r + rb' = r + \frac{\alpha}{\tau}.
\end{aligned}$$

Hence, for $0 < r \leq R$, we have

$$\begin{aligned}
g_\tau(s,t,x) &= \exp\Big\{ -\tau(\rho(t-s) - \rho(t)) + \frac{1}{4s}(y(t-s) - y(t))(2r + y(t) - y(t-s))\Big\}\\
&\leq \exp\Big(\tau s|\dot{\rho}|_\infty + |\dot{y}|_\infty(r + s|\dot{y}|_\infty)\Big) \leq e^{C(r+\tau s)} \leq Ce^{2Cr + C\frac{\alpha}{\tau}} \leq Ce^{C\frac{\alpha}{\tau}},
\end{aligned}$$

where $C = C(\mu, R)$. The same estimate for $g_\tau(\tilde{s},t,x)$ holds:

$$\max(g_\tau(s,t,x), g_\tau(\tilde{s},t,x)) \leq Ce^{C\frac{\alpha}{\tau}}.$$

We thus have

$$\begin{aligned}
&|g_\tau(s,t,x) - g_\tau(\tilde{s},t,x)| \leq \Big(\tau|\rho(t-s) - \rho(t-\tilde{s})| + \Big|\frac{1}{4s}(y(t-s) - y(t))\\
&\quad\cdot(2x - y(t-s) - y(t)) - \frac{1}{4\tilde{s}}(y(t-\tilde{s}) - y(t))(2x - y(t-\tilde{s}) - y(t))\Big|\Big)\\
&\quad\cdot\max(g_\tau(s,t,x), g_\tau(\tilde{s},t,x))\\
&\leq \Big(|\dot{\rho}|_\infty\tau(s-\tilde{s}) + \frac{1}{4}|\dot{y}|_\infty(2r + |\dot{y}|_\infty s) + \frac{1}{4}|\dot{y}|^2_\infty(s+\tilde{s})\Big)e^{C\frac{\alpha}{\tau}}\\
&\leq C(\tau(s-\tilde{s}) + r + s + \tilde{s})e^{C\frac{\alpha}{\tau}} \leq C(\frac{\alpha}{\tau} + \sqrt{\frac{2r}{\tau}}\sqrt{\alpha} + r + \frac{1}{\tau}(r + \frac{\alpha}{\tau}))e^{C\frac{\alpha}{\tau}}\\
&\leq C(\frac{\alpha}{\tau} + r)e^{C\frac{\alpha}{\tau}}, \quad C = C(\mu,R).
\end{aligned}$$

Consequently, since

$$\begin{aligned}
\varphi^-(t,x) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty (g_\tau(s,t,x) - g_\tau(\tilde{s},t,x))\frac{\sinh(a/2)}{\sinh(a)}e^{-\alpha}\frac{d\alpha}{\sqrt{\tau r}}\\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty (g_\tau(s,t,x) - g_\tau(\tilde{s},t,x))\frac{1}{2\cosh(a/2)}e^{-\alpha}\frac{d\alpha}{\sqrt{\tau r}},
\end{aligned}$$

we then have, for $\tau > C = C(\mu, R)$,

$$\begin{aligned}|\varphi^-(t,x)| &\leq C\int_0^\infty (\frac{\alpha}{\tau}+r)e^{(\frac{C}{\tau}-1)\alpha}\frac{d\alpha}{\sqrt{\tau r}\sqrt{\frac{\alpha}{2\tau r}+1}} \\ &\leq C\int_0^\infty (\frac{\sqrt{\alpha}}{\tau}+\sqrt{\frac{r}{\tau}})e^{(\frac{C}{\tau}-1)\alpha}d\alpha \\ &\leq C'(\frac{1}{\tau}+\sqrt{\frac{r}{\tau}}).\end{aligned}$$

Thus (2.13) is proved. Moreover, since

$$\begin{aligned}\nabla_x g_\tau(s,t,x) &= \frac{y(t-s)-y(t)}{2s}g_\tau(s,t,x), \\ |\nabla_x g_\tau(s,t,x)| &\leq \frac{1}{2}|\dot{y}|_\infty |g_\tau(s,t,x)| \leq Ce^{C\frac{\alpha}{\tau}},\end{aligned} \tag{2.16}$$

we then have

$$\begin{aligned}|\nabla_r\varphi^-(t,x)| &\leq C\int_0^\infty |g_\tau(s,t,x)+g_\tau(\tilde{s},t,x)|\frac{1}{2\cosh(a/2)}e^{-\alpha}\frac{d\alpha}{\sqrt{\tau r}} \\ &\leq C\int_0^\infty e^{(\frac{C}{\tau}-1)\alpha}\frac{1}{2\sqrt{\frac{\alpha}{2\tau r}+1}}\frac{d\alpha}{\sqrt{\tau r}} \\ &\leq C\int_0^\infty e^{(\frac{C}{\tau}-1)\alpha}\frac{d\alpha}{\sqrt{\alpha}} \leq C'.\end{aligned}$$

Thus the estimate of $\nabla_r\varphi^-(t,x)$ in (2.15) is proved. Let us prove (2.14). Since

$$\frac{\cosh(a/2)}{\sinh(a)} = \frac{1}{2\sinh(a/2)} = \frac{1}{\sqrt{2b'}} = \frac{\sqrt{\tau r}}{\sqrt{2\alpha}},$$

we then have

$$\varphi^+(t,x) = \frac{1}{\sqrt{2\pi}}\int_0^\infty (g_\tau(s,t,x)+g_\tau(\tilde{s},t,x))\frac{\cosh(a/2)}{\sinh(a)}e^{-\alpha}\frac{d\alpha}{\sqrt{\tau r}},$$

and so

$$|\varphi^+(t,x)| \leq C|\int_0^\infty e^{C\frac{\alpha}{\tau}-\alpha}\frac{1}{\sqrt{2\alpha}}d\alpha \leq C'.$$

This proves (2.14). Similarly, thanks to (2.16), we have

$$\begin{aligned}|\nabla_x\varphi^+(t,x)| &= \frac{1}{\sqrt{2\pi}}\left|\int_0^\infty (\nabla_x g_\tau(s,t,x)+\nabla_x g_\tau(\tilde{s},t,x))\frac{1}{\sqrt{2\alpha}}e^{-\alpha}d\alpha\right|, \\ &\leq C\int_0^\infty e^{C\frac{\alpha}{\tau}-\alpha}\frac{1}{\sqrt{2\alpha}}d\alpha \leq C',\end{aligned}$$

which proves (2.15) for $\nabla_x\varphi^+$. So Lemma 2.4 is proved.

By observing that $\varphi = \varphi^- + \varphi^+$, Lemma 1.1 is then proved.

2.3. Properties of reflected waves. Putting

$$\begin{aligned}\Psi_\tau &= (\gamma-1)\nabla_x V_\tau + \nabla_x W_\tau = \gamma\nabla_x V_\tau - \nabla_x U_\tau,\\ \Psi_j &= (\gamma-1)\nabla_x V_j + \nabla_x W_j = \gamma\nabla_x V_j - \nabla_x U_j,\end{aligned} \tag{2.17}$$

we have

$$\operatorname{div}_x\Psi_\tau = \operatorname{div}_x(\gamma(t,\cdot)\nabla_x V_\tau) - \Delta_x U_\tau = \partial_t W_\tau \quad \text{in} \quad \Omega_T, \tag{2.18}$$

$$\operatorname{div}_x\Psi_j = \operatorname{div}_x(\gamma(t,\cdot)\nabla_x V_j) - \Delta_x U_j = \partial_t W_j \quad \text{in} \quad \Omega_T. \tag{2.19}$$

If $\Sigma_T \cap \overline{D_T} = \emptyset$ then, in view of (2.17), (1.16), (1.19), we have

$$\Psi_j \overset{j\to\infty}{\to} \Psi_\tau \quad \text{in} \quad C([0,T],(L^2(\Omega))^3).$$

2.4. Proof of Lemma 1.2. Let us set $T' = T$ which does not restrict the proof. We denote U_j^* instead of $U_{j,T}^*$. Put

$$\begin{aligned}J_j(t) &:= \int_{\partial\Omega}(\partial_\nu W_j)(t,x)\,U_j^*(t,x)d\sigma(x),\\ F_j(t) &:= \int_\Omega(\gamma-1)\nabla_x V_j\nabla_x U_j^* dx.\end{aligned}$$

Thanks to (2.19) and integration by parts, we have

$$\begin{aligned}J_j(t) &= \int_{\partial\Omega}(\nu\cdot\Psi_j)U_j^* d\sigma(x) = \int_\Omega \Psi_j\nabla_x U_j^* dx + \int_\Omega(\operatorname{div}_x\Psi_j)U_j^* dx\\ &= F_j(t) + \int_\Omega(\nabla_x W_j\nabla_x U_j^* + \partial_t W_j U_j^*)dx.\end{aligned}$$

Thanks to (1.18), (1.14), we compute

$$\begin{aligned}\int_\Omega \nabla_x W_j\nabla_x U_j^* dx = -\int_\Omega W_j\Delta_x U_j^* dx &= \int_\Omega W_j(\partial_t U_j^* + \mathcal{L}_1^* U_j^*)dx\\ = \int_\Omega W_j(\partial_t U_j^* + \frac{\dot\kappa}{\kappa}U_j^*)dx &= \int_\Omega W_j\kappa^{-1}\partial_t(\kappa U_j^*)dx.\end{aligned}$$

Thus

$$J_j(t) = F_j(t) + \int_\Omega \kappa^{-1}\partial_t(W_j U_j^*\kappa)dx. \tag{2.20}$$

Thanks to (1.14), (2.20), an integration by parts according to t brings

$$I_j \equiv \int_0^T J_j(t)\kappa(t)dt = \int_0^T F_j(t)\kappa(t)dt + \int_\Omega [W_j U_j^*\kappa]_0^T dx, \tag{2.21}$$

Thanks to (1.15), (1.19), (1.21), (2.21), and to Remark 2, if $\Sigma_T\cap\overline{D_T} = \emptyset$, then we have:

$$\begin{aligned}&\int_\Omega [W_j U_j^*\kappa]_0^T dx \overset{j\to+\infty}{\to} \int_\Omega [W_\tau U_\tau^*\kappa]_0^T dx,\\ &\int_0^T F_j(t)\kappa(t)dt \overset{j\to+\infty}{\to} \tilde I_\tau \equiv \int_{\Omega_T}(\gamma-1)\nabla_x V_j\nabla_x U_j^*\kappa(t)dt,\\ &\qquad I_j \overset{j\to+\infty}{\to} I_\infty(\tau,\mu,\theta,T) \equiv \tilde I_\tau + \int_\Omega [W_\tau U_\tau^*\kappa]_0^T dx,\end{aligned}$$

which proves (1.22) in (1) of Lemma 1.2.

2.5. Proof of Theorem 1. For the sake of simplicity, we assume that $\gamma \geq 1$. The case $\gamma \leq 1$ is similar. We put

$$F_\tau := \int_\Omega (\gamma - 1)\nabla_x V_\tau \nabla_x U_\tau^* dx.$$

2.5.1. Lower Bound for I_∞. Let us give another expression of

$$\tilde{I}_\tau = \int_0^T F_\tau(t,x)\kappa(t)dt.$$

We remind that

$$U_\tau^* = \rho e^{-2\tau^2 t} U_\tau, \quad \rho := \frac{\varphi^*}{\varphi}, \tag{2.22}$$

where φ, φ^* are characterized in Lemma 1.1, and we put

$$\begin{aligned} r_1 &:= \frac{\gamma-1}{\gamma}\Psi_\tau(\frac{\nabla_x \rho}{\rho})U_\tau^*, \\ r_2 &:= \frac{\gamma-1}{\gamma}\Big\{(\nabla_x\varphi^*\varphi + \varphi^*\nabla_x\varphi)p_{\tau,\Sigma}\nabla_x p_{\tau,\Sigma} + \nabla_x\varphi^*\nabla_x\varphi|p_{\tau,\Sigma}|^2\Big\}. \end{aligned}$$

We firstly have

$$\begin{aligned} (\gamma-1)\nabla_x V_\tau \nabla_x U_\tau^* &\overset{(1)}{=} \frac{\gamma-1}{\gamma}\Psi_\tau\nabla_x U_\tau^* + \frac{\gamma-1}{\gamma}\nabla_x U_\tau \nabla_x U_\tau^* \\ &\overset{(2)}{=} r_1 + \frac{\gamma-1}{\gamma}\Psi_\tau\rho\nabla_x U_\tau e^{-2\tau^2 t} + \frac{\gamma-1}{\gamma}\nabla_x U_\tau\nabla_x U_\tau^* \\ &\overset{(3)}{=} r_1 + \frac{1}{\gamma}\Psi_\tau^2\rho e^{-2\tau^2 t} - \Psi_\tau(\nabla_x W_\tau)\rho e^{-2\tau^2 t} + \frac{\gamma-1}{\gamma}\nabla_x U_\tau \nabla_x U_\tau^* \\ &= r_1 + \frac{1}{\gamma}\Psi_\tau^2\rho e^{-2\tau^2 t} - \Psi_\tau(\nabla_x W_\tau)\rho e^{-2\tau^2 t} + \frac{\gamma-1}{\gamma}|\nabla_x p_{\tau,\Sigma}|^2\varphi^*\varphi + r_2. \end{aligned}$$

Explanations - (1): write $\nabla_x V_\tau = \frac{1}{\gamma}(\Psi + \nabla_x U_\tau)$. (2): use (2.22). (3): write $(\gamma-1)\nabla_x U_\tau = \Psi_\tau - \gamma\nabla_x W_\tau$ in the second term.

By integration by parts, thanks to (1.20) and (2.18), we have

$$\begin{aligned} -\int_\Omega \Psi_\tau \nabla_x W_\tau \rho dx &= \int_\Omega \partial_t W_\tau W_\tau \rho dx + r_3, \\ r_3 &:= \int_\Omega \Psi_\tau W_\tau \nabla_x \rho dx. \end{aligned} \tag{2.23}$$

We thus have

$$\begin{aligned} F_\tau = &\int_\Omega \frac{\gamma-1}{\gamma}|\nabla_x p_{\tau,\Sigma}|^2\varphi^*\varphi e^{-2\tau^2 t}dx + \int_\Omega \frac{1}{\gamma}\Psi_\tau^2\rho e^{-2\tau^2 t}dx \\ &+ \int_\Omega (\partial_t W_\tau)W_\tau\rho e^{-2\tau t^2}dx + \int_\Omega (\frac{\gamma-1}{\gamma}(r_1+r_2)+r_3)dx. \end{aligned}$$

Writing

$$\begin{aligned} \int_{\Omega_T}(\partial_t W_\tau)W_\tau\rho e^{-2\tau t^2}\kappa dtdx = &\int_{\Omega_T}(\tau^2\rho - \frac{1}{2}\dot{\rho})|W_\tau|^2 e^{-2\tau t^2}\kappa dtdx \\ &+\frac{1}{2}\int_\Omega [W_\tau^2 e^{-2\tau t^2}\kappa]_0^T, \end{aligned} \tag{2.24}$$

we then have

$$
\begin{aligned}
I_\infty(\tau,\mu,\theta,T) &= \tilde{I}_\tau + \int_\Omega [W_\tau U_\tau^* \kappa]_0^T dx \\
&= \int_{\Omega_T} \frac{\gamma-1}{\gamma} |\nabla_x p_{\tau,\Sigma}|^2 \varphi^* \varphi \kappa dt dx + \int_{\Omega_T} \frac{1}{\gamma} \Psi_\tau^2 \rho e^{-2\tau^2 t} \kappa dt dx \\
&\quad + \int_{\Omega_T} (\tau^2 \rho - \frac{1}{2}\dot{\rho}) |W_\tau|^2 e^{-2\tau t^2} \kappa dt dx + \frac{1}{2} \int_\Omega (W_\tau^2 e^{-2\tau t^2}) \kappa|_{t=T} dx + R,
\end{aligned} \tag{2.25}
$$

where we put

$$
\begin{aligned}
R &:= R_1 + R_2 + R_3 + R_4 + R_5, \\
R_j &:= \int_{\Omega_T} r_j \kappa dt dx, \quad j = 1,2,3, \\
R_4 &:= \int_\Omega [W_\tau U_\tau^* \kappa]_0^T dx, \\
R_5 &:= -\frac{1}{2} \int_\Omega (W_\tau^2 \kappa)|_{t=0} dx,
\end{aligned}
$$

We prove in the next Lemma that R is neglictable.

Lemma 2.5. *There exist $\delta > 0$, $\mu_1 > 0$, so that for all $\mu > \mu_1$, there exists $\tau_1 > 0$ so that for all $\tau > \tau_1$, the following estimates hold.*

$$|R_1| \le \tau^{-1} \int_{\Omega_T} \Psi_\tau^2 \rho e^{-2\tau^2 t} \kappa dt dx + C\tau^{-1} \int_{\Omega_T} |\gamma - 1| \, |\nabla_x p_{\tau,\Sigma}|^2 \kappa dt dx; \tag{2.26}$$

$$|R_2| \le C\tau^{-1} \int_{\Omega_T} |\gamma - 1| \, |\nabla_x p_{\tau,\Sigma}|^2 \kappa dt dx; \tag{2.27}$$

$$|R_3| \le \int_{\Omega_T} \frac{1}{2\gamma} \Psi_\tau^2 \rho e^{-2\tau^2 t} \kappa dt dx + C \int_{\Omega_T} |W_\tau|^2 e^{-2\tau^2 t} \kappa dt dx; \tag{2.28}$$

$$|R_4| \le \frac{1}{4} \int_\Omega |W_\tau|^2|_{t=T} e^{-2\tau T} \kappa(T) dx + Ce^{-2\tau(y(t),D(t))+\delta)}, \tag{2.29}$$

$$|R_5| \le Ce^{-2\tau(y(t),D(t))+\delta)}, \quad t \in [0,T]. \tag{2.30}$$

Proof. a) Since $p_{\tau,\Sigma} \le \tau |\nabla_x p_{\tau,\Sigma}|$ and thanks to Remark 1, we obtain (2.26) and (2.27).
b) Estimates (2.28) comes from (1.10), (1.11), from Remark 1 and standart inequalities.
c) Thanks to Lemma 1.1, (1.10), to Remark 1, to (2.1), we have

$$|R_4| \le \frac{1}{4} \int_\Omega |W_\tau|^2|_{t=T} e^{-2\tau T} \kappa(T)) dx + C \int_\Omega |p_{\tau,y(T)}|^2 \kappa(T) dx + \int_\Omega |W_\tau U_\tau^*|_{t=0} \kappa(0) dx$$

d) Estimate (2.30) is (2.8). Then (2.29) comes from (2.6) , (2.9). This proves the Lemma.

From Lemma 2.5, (2.25), (1.10), Remark 1, and since

$$\tau^2 \rho - \frac{1}{2}\dot{\rho} + C \ge C' \tau^2, \quad C' > 0,$$

for τ sufficiently large, we obtain

$$I_\infty(\tau,\mu,\theta,T) \geq \frac{1}{C}\Big\{ \int_{\Omega_T} \frac{\gamma-1}{\gamma}|\nabla_x p_{\tau,\Sigma}|^2 \kappa dtdx + \int_{\Omega_T} \Psi_\tau^2 \rho e^{-2\tau^2 t}\kappa dtdx \\ + \int_{\Omega_T} \tau^2 |W_\tau|^2 e^{-2\tau t^2}\kappa dtdx + \int_\Omega |W_\tau|^2|_{t=T} e^{-2\tau T^2}\kappa(T)dx \Big\},$$

for some $C>1$, which implies the lower bound for $|I_\infty|$ in (1.23).

2.5.2. Upper Bound for I_∞. In the same way, putting

$$r_6 := (\gamma-1)\nabla_x W_\tau (\nabla_x \rho) U_\tau e^{-2\tau^2 t},$$

we have,

$$\begin{aligned}(\gamma-1)\nabla_x V_\tau \nabla_x U_\tau^* &\overset{(1)}{=} (\gamma-1)\nabla_x W_\tau \nabla_x U_\tau^* + (\gamma-1)\nabla_x U_\tau \nabla_x U_\tau^* \\ &\overset{(2)}{=} (\gamma-1)\rho \nabla_x W_\tau \nabla_x U_\tau e^{-2\tau^2 t} + (\gamma-1)\nabla_x U_\tau \nabla_x U_\tau^* + r_6 \\ &= \gamma\rho\nabla_x W_\tau \nabla_x U_\tau e^{-2\tau^2 t} - \rho\nabla_x W_\tau \nabla_x U_\tau e^{-2\tau^2 t} + (\gamma-1)\nabla_x U_\tau \nabla_x U_\tau^* + r_6 \\ &\overset{(3)}{=} -\gamma\rho\nabla_x |W_\tau|^2 e^{-2\tau^2 t} + \rho\nabla_x W_\tau \Psi_\tau e^{-2\tau^2 t} + (\gamma-1)\nabla_x U_\tau \nabla_x U_\tau^* + r_6.\end{aligned}$$

Explanations - (1): write $V_\tau = U_\tau + W_\tau$. (2): use (2.22). (3): write $U_\tau = V_\tau - W_\tau$ in the firs two terms.

Thanks to (2.23), we have

$$F_\tau = -\int_\Omega \gamma\rho\nabla_x |W_\tau|^2 e^{-2\tau^2 t}dx - \int_\Omega \partial_t W_\tau W_\tau \rho dx - r_3 \\ + \int_\Omega (\gamma-1)\nabla_x U_\tau \nabla_x U_\tau^* dx + \int_\Omega r_6 dx.$$

Thanks to (2.23), (2.24), we obtain

$$\tilde{I}_\tau = -\int_{\Omega_T} \gamma\rho|\nabla_x W_\tau|^2 e^{-2\tau^2 t}\kappa dtdx - \int_{\Omega_T} (\tau^2\rho - \frac{1}{2}\dot{\rho})|W_\tau|^2 e^{-2\tau t^2}\kappa dtdx \\ -\frac{1}{2}\int_\Omega (W_\tau^2 e^{-2\tau t^2}\kappa)_{t=T}dx + \int_{\Omega_T} (\gamma-1)\nabla_x U_\tau \nabla_x U_\tau^* \kappa dtdx \\ -R_3 - R_5 + R_6,$$

where we put

$$R_6 := \int_{\Omega_T} r_6 \kappa(t) dtdx.$$

Similarly to the above section §2.5.1, for τ sufficiently large, we obtain

$$I_\infty(\tau,\mu,\theta,T) \leq \frac{1}{C}\Big\{ \int_{\Omega_T} (\gamma-1)|\nabla_x p_{\tau,\Sigma}|^2 \kappa dtdx - \int_{\Omega_T} |\nabla_x W_\tau|^2 e^{-2\tau^2 t}\kappa dtdx \\ - \int_{\Omega_T} \tau^2 |W_\tau|^2 e^{-2\tau t^2}\kappa dtdx - \int_\Omega |W_\tau|^2|_{t=T} e^{-2\tau T^2}\kappa(T)dx \Big\},$$

for some $C>1$, which implies the upper bound for $|I_\infty|$ in (1.23).

Thus Theorem 1 is proved.

2.6. Proof of Theorem 2. a) Assume that $T < T^*$. Let $\theta \in (0,T)$ and $\mu > \mu_1$, where μ_1 is defined in Theorem 1. Thanks to (2.5) and to Theorem 1, we obtain (1.24). Since in this formula the right-hand side does not depend on μ, we let μ tend to infinity. We then obtain (1.25).

b) Assume that $\Sigma_T \cap \overline{D_T} \neq \emptyset$, that is, $T \geq T^*$. Let $0 < T' < T^*$. Thus, (1.25) holds with T replaced by T'. Taking the limit as $T' \nearrow T^*$, and since $\mathrm{d}(\Sigma_{T'}, D_{T'})$ tends to $\mathrm{d}(\Sigma_{T^*}, D_{T^*}) = 0$ when, we obtain (1.26).

2.7. Proof of Lemma 1.3. First remind that F is defined at least for T' sufficiently close to 0. Observe that the function G:

$$G: \quad [0,T] \supset T' \mapsto -2\mathrm{d}(\Sigma_{T'}, D_{T'})$$

is lipschitzian, non-decreasing, and $G(0) < 0$. Thus, there exists $\delta > 0$ such that

$$G(T'') \leq G(T') + 2\delta(T'' - T'), \quad 0 \leq T' \leq T'' \leq T.$$

Hence, if $G(T') < 0$, then $G(T' + |G(T')|/\delta) < 0$. The result is proved since , $G = F$ on $[0, T^*[$.

2.8. Existence of Needle sequences. Denote by Σ^C the open set $((-1, T+1) \times \Omega') \backslash \Sigma_{[-1,T+1]}$, with $\Sigma_{[t,t']} = \{(s, y(s));\, t \leq s \leq t'\}$. Let us prove the existence of $\{u_j\}_j$. Let $G(t,x)$ be the solution in $L^2((0,T+1); H_0^1(\Omega')) \cap H^1((0,T+1); L^2(\Omega')) \subset C([-1,T+1]; L^2(\Omega'))$ of

$$\begin{aligned} (\mathcal{L}_1 + \tau\eta)G = 0 \quad &in \quad \Omega'_{T+1}, \\ G|_{t=0} = U_\tau(0,\cdot) \quad &in \quad \Omega', \end{aligned}$$

and $\partial_t G \in L^2((0,T+1); H^{-1}(\Omega'))$. Since $U_\tau \in C([-1,T+1]; L^2(\Omega'))$ and $\partial_t U_\tau \in L^\infty((-1,T+1); H^{-1}(\Omega'))$, then the function $H = U_\tau - G$ extended by 0 for $t < 0$ belongs to $C([-1,T+1]; L^2(\Omega'))$ with $\partial_t H \in L^2((-1,T+1); H^{-1}(\Omega'))$, and satisfies

$$\begin{aligned} (\mathcal{L}_1 + \tau\eta)H\,(t,x) = \chi_{(0,+\infty)}(t) \otimes \delta(x - y(t)) \quad &in \quad (-1,T+1) \times \Omega', \\ H|_{t \leq 0} = 0 \quad &in \quad \Omega'. \end{aligned}$$

Let U be a simply connected open set in $\mathbb{R}^4$ with lipschitz boundary, and such that $\overline{U} \subset \Sigma^C$. Observe that $\Sigma^C \setminus \overline{U}$ is connected. For all open set V such that $\overline{U} \subset V \subset \overline{V} \subset \Sigma^C$, we have $H|_V \in H^{1,2}(V)$, $H|_V(t,\cdot) = 0$ for $-1 < t \leq 0$, and $(\mathcal{L}_1 + \tau\eta)H = 0$ in V. Similarly to [2, Theorem 2.3], there exists a sequence $H_j \in (-1,T+1) \times \Omega'$ such that

$$\begin{aligned} (\mathcal{L}_1 + \tau\eta)H_j\,(t,x) = 0 \quad &\text{in} \quad (-1,T+1) \times \Omega', \\ H_j|_{t \leq 0} = 0 \quad &\text{in} \quad \Omega', \end{aligned} \tag{2.31}$$

and H_j converges to H in $L^2(U)$ (with the strong norm). We then have the existence of another sequence $H'_j \in H^{1,2}_{loc}(\Sigma^C)$ satisfying the homogeneous relation (2.31) and such that H'_j converges to H in $H^{1,2}_{loc}(\Sigma^C)$, as noted in [2, §3]. Finally, the sequence $U_j = H_j + G$ satisfies Equation (2.31) in Ω'_T, with $U_j|_{t=0} = U_\tau(0)$ in Ω', and $U_j \to U_\tau$ in $H^{1,0}_{loc}(\Omega_T \setminus \Sigma_T) \cap H^{0,1}_{loc}(\Omega_T \setminus \Sigma_T)$.

REMARK 6. *We used the fact that $U_\tau(0) \in L^2(\Omega')$ only. Hence, the proof for the existence of U_j^* in the dual case, where the triplet $(\mathcal{L}_1, t, t=0)$ is replaced by $(\mathcal{L}_1^*, T-t, t=T)$, works similarly.*

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